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Manoff's generalized deviation equation and its possible applications in celestial mechanics and relativistic astrometry

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This report proposes a relationship between celestial mechanics and gravitational theory on the base of the s.c. "generalized deviation equation" (GDE), proposed for the first time by Prof. S. Manoff. Using the Hill-Brown system of equations in the framework of the restricted three-body problem Earth-Moon-Sun, it has been proved that it is possible to find the metric tensor components of the gravitational field for the case of stationary, diagonal metric and in the vicinity of the libration points in the space between the Earth and the Moon. Possible applications of the GDE in relativistic astrometry are briefly discussed based on the property of the GDE to account for the relative acceleration and for non-infinitesimal trajectory deviations, caused by the curvature of the space-time geometry.

Key words: celestial mechanics (including n-body problems), astrometry and reference systems, general relativity and gravitation (fundamental problems and general formalism), modified theories of gravity

THE MATHEMATICAL LIMITATIONS OF THE GEODESIC AND GEODESIC DEVIATION EQUATIONS IN GENERAL RELATIVITY

Contemporary celestial mechanics and (Einsteinian) gravitational theory both treat one and the same physical object – the gravitational field around celestial bodies. However, the theoretical grounds of these two theories are different – celestial mechanics is based on the Newton's equation $m\ddot{r} = F_{ext.}$, while the central notion of Einstein's theory is based on the metric tensor $g_{\mu\nu}$ of the gravitational field and the notion of metric $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$.

The implementation of the variational principle in Einstein's gravity with respect to a length functional allows one to derive the geodesic equation

$$\frac{Du^{\mu}}{Dt} = \frac{dx^{\mu}}{dt} + \Gamma^{\mu}_{\lambda\rho}u^{\lambda}u^{\rho} = 0\,. \qquad (1)$$

It is important to mention that the variational formalism (see [1]) is performed for proper variations (i.e. with fixed endpoints). Then γ will be stationary for the length functional and for the energy functional if γ is parametrized proportionally to arclength (meaning $\parallel \dot{\gamma}(t,0) \parallel = const.$).

Now if X is a vector field along the geodesic line $\gamma : [a,b] \rightarrow M$ and Y is a vector field along the same geodesic line, but with fixed endpoints, i.e. $Y(a) = Y(b) = 0$, then the vector field X is called a *Jacobi field* if it satisfies the equation

$$\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}X + R(\dot{\gamma},X)\dot{\gamma} = 0. \qquad (2)$$

The Jacobi equation (2) is a modified, but more general version of a geodesic deviation equation. Remarkably, in [2] the equation (2) was shown to follow from the standard definition for the *curvature operator* $R(X,Y)$

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z = R(X,Y)Z \qquad (3)$$

without the application of any variational principle and under the following more general assumptions:

1. the trajectories of the vector field X are affinely parametrized autoparallels, i. e. $\nabla_X X = 0$. These autoparallel lines have no fixed endpoints. The physical meaning of the autoparallels is that along these lines a test particle would not experience any relative acceleration.

2. the vector fields X and Y commute, i.e. $\pounds_X Y = [X,Y] = 0$.

3. torsion is assumed to vanish $T(X,Y) = 0$, which with account of the formulae

$$\pounds_X Y = \nabla_X Y - \nabla_Y X - T(X,Y) \qquad (4)$$

and the preceeding assumption, gives the equality $\nabla_X Y = \nabla_Y X$. The autoparallel line $\nabla_X X = 0$ is a more general notion compared to the geodesic line, since the autoparallel line is not necessarily a geodesic line. The main motivation for searching new types of deviation equations is the increasing accuracy (of the order of 1 μas - 1 microarcsecond and even less) for the future astrometric measurements for space missions.

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This Jacobi equation in the paper [2] is called *"the equation of deviation of autoparallels", since it describes geodesics, close to the reference geodesic and with velocities, close to the velocity of the reference geodesic.*

The linearization procedure and the closeness to the reference geodesic is fully consistent with the assumptions for zero relative acceleration $\nabla_X X = 0$ and zero Lie-derivative $\pounds_X Y = [X,Y] = 0$.

More interesting, however, is the case of non-zero Lie-derivative $\pounds_X Y \neq 0$. In such a case, in [3] it has been shown that the "infinitesimality" concept breaks down due to the "moving neighbourhoods" along the curve. In any case, for nonzero relative acceleration and Lie-derivative the deviation equation is not known.

THE MAIN OBJECTIVES OF THIS PAPER

The purposes of this paper is to discuss briefly and provide the motivation for possible applications of the "generalized deviation equation" (GDE) in relativistic astrometry and celestial mechanics. This equation has been proposed for the first time in 1999 by Prof. Sawa Manoff in the review paper [6]. Based on the unusual properties of this equation, related to non-infinitesimal deviations, a concrete application is considered, related to the libration points in the restricted three-body problem Earth-Moon-Sun. The application is based on using the unique solution of the linearized version of the Hill-Brown system of equations (see the known monograph [7]) for the description of the deviation of a single trajectory, which is given by the GDE for the case of coinciding trajectories. Thus, from the Hill-Brown system and the GDE, the metric field components can be found.

RELATIVISTIC ASTROMETRY AND SOME ASTROPHYSICAL MOTIVATION FOR SEARCHING NEW DEVIATION EQUATIONS IN GRAVITY THEORY

Relativistic astrometry investigates light propagation and light deflection from large celestial bodies, taking into account the action of the gravitational field. After perturbing the (flat Minkowski) gravitational background as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the light trajectory deviates slightly [8] and is described by the null geodesic equation in a weak gravitational field.

Such an approach is applied in the theoretical modelling of the relativistic *VLBI* experiment on September the 8, 2002 [9]. On 8 September, 2002 Jupiter passed within $3''.7$ ($\sim$ 14 Jovian radii) of the bright, distant radio quasar $J0842+1835$. The Jovian experiment enabled the measurement of the speed of gravity $c_g = (1.06 \pm 0.2)c$ and the additional retardation of the light signal due to the dragging of the light ray, caused by the time-dependent gravitational field, generated by the translational motion of Jupiter. The dependence on the translational speed v appears after the retarded velocity solution of the Einstein's equations is substituted into the solution of the light geodesic equation [10], where the first term is related with the unperturbed light ray trajectory - the straight line $x^i(t) = x^i_0 + k^i(t - t_0)$. However, the light geodesic equation by itself does not possess the property to account for light deflection by moving sources. Such a possibility exists for the Manoff's generalized deviation equation (GDE), since it can give the deviating trajectory with respect to two other vector fields (tangent to two non-infinitesimally deviating trajectories), as it will be shown in the next section. Evidently, the combination of the Manoff's GDE with the solution of the Einstein's equations would give higher order $\frac{v}{c}$ deflection terms. This is important, since the formalism should not be limited to first-order $\frac{v}{c}$ terms, as in [10]. The reason is that in future astrometric missions like *GAIA* and *SIM* (Space Interferometry Mission), the relativistic deflection terms of order $\frac{v^2}{c^2}$ may reach 1 μas and can be in principle detected [11].

The hypothesis about "isolateness" is not fulfilled, since the participation of the Solar system in the process of the global (Hubble) cosmological expansion [13] results in the fact that the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric defines the properties of space-time not only on the scale of galaxy clusters, but also within the Solar system [14]. The local Hubble expansion [15] is equivalent to a deviation from inertial motion (if understood in the Newtonian sense), given with the Jacobi type of a equation $\frac{d^2x^k}{d\tau^2} + R^k_{0l0}x^k = 0$. This Newtonian type of an equation, as previously explained, is obtained for the case of a zero relative acceleration.

For the case of a non-zero relative acceleration, the usual geodesic equation will be inapplicable, since its mathematical derivation requires fixed end-points of the geodesic line. From the simple formulae $l = g(u,u)$ for the length of the vector field u after covariant differentiation it can be seen that the change of length is related to the non-zero relative acceleration a. On the other hand, a perturbed theory on a curved FLRW background field would lead to the s.c. "anisotropic scale factor" [4], since $g_{\mu\nu}g^{\alpha\nu} = (g^{(0)}_{\mu\nu} + h_{\mu\nu})(g^{(0)\alpha\nu} - h^{\alpha\nu}) = \delta^{\alpha}_{\mu} - h_{\mu\nu}h^{\alpha\nu} = l^{\alpha}_{\mu}$. This would lead to a non-zero relative acceleration. So again, the Generalized Deviation Equation would account for deviations of the trajectories of motion.

An example about non-infinitesimall trajectory deviations is given in the monograph of Seeber [5] on satellite geodesy, where several reasons are pointed out for the perturbation of the orbit of a *GPS*-satellite: due to the Earth's obliquity and the nonhomogeneous distribution of matter (inside the Earth and on its surface), the satellite can have an acceleration of $5 \times 10^{-5}\ m/s^2$, causing a deviation of 2 *km* from the orbit in the course of 2 hours. Other celestial bodies such as the Sun, the Moon cause an additional (tidal) acceleration of $5 \times 10^{-6}\ m/s^2$, leading a deviation of the orbit from 5 to 150 meters again for 2 hours. The last acceleration is 10% of the first one, so it is problematic whether deviating satellite trajectories can be considered as perturbations and infinitesimal deviations.

THE NEW UNUSUAL FEATURES OF THE GENERALIZED DEVIATION EQUATION AND ITS MATHEMATICAL DERIVATION

This section has the purpose to present briefly the mathematical derivation of the GDE and to point out some unusual features of this equation. A central initial moment in Manoff's derivation of the GDE in the paper [6] is the formulae for the Riemann operator

$$[R(\xi,u)] = \nabla_{\xi}\nabla_{u} - \nabla_{u}\nabla_{\xi} - \nabla_{£_{\xi}u}\,, \qquad (5)$$

which can act on a given vector or tensor field. Then the commutator $[\nabla_w, R(\xi,u)]$ ($w,\xi,u \in T(M)$ - the tangent space to the manifold) can be obtained in the following form

$$[\nabla_w, R(\xi,u)] = [\nabla_w, £\Gamma(\xi,u)] + [\nabla_w,[\nabla_{\xi},\nabla_u]] - [\nabla_w,[£_{\xi},\nabla_u]]. \qquad (6)$$

It is important to note that neither of the three vector fields w,ξ,u is assumed to be a geodesic line. A new operator has appeared in the above theoretical scheme - *the deviation operator* $£\Gamma(\xi,u)$:

$$[£\Gamma(\xi,u)]v = [R(\xi,u)]v + [\nabla_u\nabla_v - \nabla_{\nabla_u v}]\xi - T(\xi,\nabla_u v) + \nabla_u[T(\xi,v)], \qquad (7)$$

where $T(\xi,v)$ is the torsion operator, which will be neglected further. After calculating the commutator

$$[\nabla_w, £\Gamma(\xi,u)] \equiv [\nabla_w,[£_{\xi},\nabla_u]] - R(w,£_{\xi}u),$$

the expression for the deviation operator $[£\Gamma(\xi,u)]v$ can be rewritten as

$$\nabla_u\nabla_v\xi = ([R(u,\xi)]v) + \nabla_{\xi}\nabla_u v - \nabla_u £_{\xi} v - \nabla_{£_{\xi}u} v - \nabla_u[T(\xi,v)]. \qquad (8)$$

This is the generalized deviation equation (GDE), proposed in 1999 by Prof. S. Manoff in [6]. This equation possess the following properties, quite different from the known equation:

1. The equation gives the deviation (vector field) ξ with respect to two other vector fields u and v. Neither of the trajectories, to which these vector fields are tangent, is assumed to be a geodesic line. This is one of the key arguments for the application of the GDE in celestial mechanics.

2. Equation (8) is written in terms of covariant derivatives, which are defined for non-infinitesimal transports of a given vector field along a contour. This equation gives also the non-infinitesimal deviation of the vector field ξ from the two other fields u and v. But this means also that each one of these fields can also be considered to be a deviation field. In other words, the GDE can be written also for the case of the cyclic transformations $\xi \to u \to v \to \xi$ and also $\xi \to v \to u \to \xi$. Thus, a configuration of three trajectories in a gravitational field can be considered to be a stable one, described by a system of three GDEs.

3. The physical reason for the non-infinitesimal deviations is the action of the relative acceleration. This can be seen from an earlierly proposed (again by S. Manoff in the paper [16]) version of the GDE

$$\frac{D^2\xi^i}{ds^2} = R^i_{klj}u^k u^l \xi^j + \xi^i_{;j}F^j + u^k u^l £_{\xi}\Gamma^i_{kl} = u^k u^l \xi^n R^i_{nlk} + u^k\left[\left(\xi^i_{;l}u^l\right)_{;k}\right], \qquad (9)$$

where the Lie-derivative $£_{\xi}\Gamma^i_{kl} = \xi^i_{;k;l} - R^i_{klj}\xi^j$ and $a = F = \frac{Du}{d\tau} = u^{\alpha}\nabla_{\alpha}u$ is the relative acceleration. This equation differs from the proposed by J. L. Synge equation of geodesic deviation $\frac{D^2\xi^i}{ds^2} = u^k u^l \xi^n R^i_{nlk}$ only by the last term $u^k\left[\left(\xi^i_{;l}u^l\right)_{;k}\right]$, related to the second covariant derivative $\nabla_u\nabla_u\xi = (\xi^i{}_{;l}\cdot u^l)_{;m}\cdot u^m\cdot e_i$ of the vector field ξ along the vector field u. In fact, the *orthogonal to a non-isotropic vector field u ($g(u,u) = e \neq 0$) projection of this second covariant* *derivative, taken along the same non-isotropic vector field u*, represents the relative acceleration $_{rel}a = \overline{g}(p_u(\nabla_u \nabla_u \xi))$.

The earlier version of the GDE (9) is derived under the restrictive assumptions $F^i u_i = 0$, $u_i u^i = e$, $e^2 = 1$, the first of which means that the relative acceleration is assumed to be perpendicular to the velocity and the second one is related to the normalization of the velocity vector field. The perpendicularity of the velocity vector and the relative acceleration is characteristic for shear- and expansion- free flows [3]. However, the more general equation (8) is valid for flows with shear and expansion.

The relative acceleration turns out to be of crucial importance for the GDE. Previously, a number of different geodesic deviation equations have been obtained by Synge and Schild, by Mashhoon, by Maugin. In Manoff's earlier paper [16] it was proved that by specifying the choice for the Lie-operator, all these different equations can be derived from equation (9). Most importantly, a solution has been found for the deviation vector describing non-infinitesimally deviating trajectories.

EARLIER ATTEMPTS FOR APPLICATION OF THE GEODESIC DEVIATION EQUATION IN CELESTIAL MECHANICS

An attempt to investigate the perturations of a geodesic trajectory in the gravitational field of a Schwarzschild Black Hole has been undertaken by R. Kerner in [18]. From the Jacobi equation $\nabla_Z^2([X,Y]) = R(Z,[X,Y])Z$ after assuming a zero relative acceleration $\nabla_Z Z = 0$ and also that X and Y are *two linearly independent Jacobi fields*, satisfying $[X,Z] = [Y,Z] = 0$, an (inhomogeneous) extension of the first-order geodesic deviation equation has been obtained. From the found solution (see also [19]), after developing into power series in the eccentricity e, the Kepler's law from celestial mechanics has been obtained:

$$r(t) = \frac{a(1-e^2)}{1+e\cos(\omega_0 t)} \simeq a(1-e\cos\omega_0 t)\,, \qquad (10)$$

where $\omega_0 = \frac{l}{a^2}$ and a is the greater half-axis. So the problem appears: what celestial mechanics trajectories can be found when the relative acceleration is not zero?

For the case in this paper, the trajectory will be found from the condition $\nabla_u \nabla_u u = 0$ for the non-geodesic deviation of a single trajectory, which however does not mean that the relative acceleration $\nabla_u u$ is zero. Also, no expansion in small eccentricity will be used.

APPLICATION OF MANOFF'S GENERALIZED DEVIATION EQUATION IN THE RESTRICTED THREE-BODY PROBLEM EARTH–MOON–SUN IN CELESTIAL MECHANICS

In this section the s.c. "inverse problem" for celestial mechanics will be solved - finding the metric field components if certain celestial mechanics trajectories are known for some specific case. In the present case, the trajectories will be taken from the known Hill-Brown system of equations, describing the motion of a material (not extended) body in the gravitational field of the three-body system Earth-Moon-Sun [20]

$$\frac{d^2x}{dt^2} - 2\frac{d\lambda'}{dt}\frac{dy}{dt} - \frac{d^2\lambda'}{dt^2}y = \frac{dU}{dx}\,, \qquad (11)$$

$$\frac{d^2y}{dt^2} + 2\frac{d\lambda'}{dt}\frac{dx}{dt} + \frac{d^2\lambda'}{dt^2}x = \frac{dU}{dy}\,, \qquad (12)$$

$$\frac{d^2z}{dt^2} = \frac{dU}{dz}\,, \qquad (13)$$

where U is the *potential function*

$$U = \frac{\mu}{\sqrt{x^2+y^2+z^2}} + \frac{n'a'^3}{\sqrt{(r'-x)^2+y^2+z^2}} - \frac{n'^2a'^3}{r'^2}\,. \qquad (14)$$

In the above formulaes, n' is the *mean angular velocity* of the Sun about the Earth, μ is the *sum of the Earth's and the Moon's masses.* In the original paper by Hill [20], another simplification is assumed - the solar eccentricity is neglected, which means that the mean angular velocity of the Sun is considered to be constant, i.e.

$$\frac{d\lambda'}{dt} = n'\,, \quad \frac{d^2\lambda'}{dt^2} = 0, \quad r' = a'\,. \qquad (15)$$

In the monograph [7] this system was analyzed in the aproximation of small deviation (X,Y) from the points (x_0,y_0), i.e. $x = x_0 + X$, $y = y_0 + Y$, where the points (x_0,y_0) are chosen to be non-moving in a reference system, rotating with the mean angular velocity n'.These are *the points of libration*, at which the potential energy has a minimum, i.e. $\frac{dU}{dx} = \frac{dU}{dy} = 0$.

Around the libration points, the Hill's system of equations in a linearized approximation can be presented in the form

$$\dot{X} = AX, \tag{16}$$

where X and A are the corresponding vector and 4×4 matrix respectively:

$$X^T := \left(X, A, \dot{X}, \dot{Y}\right), \tag{17}$$

$$A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & 2 & 0 \end{pmatrix}. \tag{18}$$

Since the libration points (x_0, y_0) represent equilibrium (stationary) values, which can be determined uniquely and the solutions (19) for X and Y are also unique (and dependent on time) [7], it can be assumed that the three vector fields u, v and ξ in the generalized deviation equation (8) will coincide and can be determined in the following way

$$\xi = u = v = \left(u_0, \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t}, \frac{\partial Z}{\partial t}\right), \tag{19}$$

where u_0 will be the $t-$coordinate value of the vector field u, not specified by the Hill's system of equations - further it will be quite natural to put $u_0 = 0$.

Further if $u = v = \xi$ is put in the *generalized deviation equation* (8), then all the terms with the exception of the first on the left - hand side will be equal to zero and so one will be left only with the first term

$$\nabla_u \nabla_u u = 0, \tag{20}$$

which is the analogue of the first term $\frac{d^2x}{ds^2}$ in the usual geodesic deviation equation.

SOLUTIONS OF THE MANOFF'S GENERALIZED DEVIATION EQUATION FOR THE LINEARIZED CASE (NEAR THE LIBRATION POINTS) OF THE HILL-BROWN SYSTEM

We shall proceed with the representation of the equation $\nabla_u \nabla_u u = 0$ for the case of three coinciding solutions $u = v = \xi$ of the *linearized Hill's system of equations in the vicinity of the libration points in the form of a nonlinear system of equations with respect to the (background) metric tensor components.* It shall be assumed that these components are only diagonal.

Since the Hill-Brown system is formulated for the three-dimensional case (three components of the vector field u) and the system of deviation equations is formulated for the four-dimensional case, we shall make the additional assumption that the zero-velocity component u_0 is zero. Further, assuming again that the Greek letters α, β, γ, $\delta = 0,1,2,3$ are the four-dimensional indices and the Latin ones i, j, k, l - the three-dimensional ones, the following equation can be obtained for the $\alpha = 0$ component and for the stationary case $\dot{X} = \dot{Y}_l = 0$:

$$u_l u_{m,l} e^{-Y_l} X_{,m} + \frac{1}{2} u_l e^{-Y_m} [u_{[m} X_{,l]} Y_{l,m} - u_m X_{,m} Y_{m,l}] - \frac{1}{2} (u_l)^2 e^{-Y_l} X_{,l} Y_{l,l} = 0, \tag{21}$$

where the following variables have been introduced:

$$\begin{aligned} X &:= \ln g_{00}, \\ Y_l &:= \ln g_{ll}, \\ Z_l &:= \frac{g_{00}}{g_{ll}} = \exp\left(X - Y_l\right). \end{aligned} \tag{22}$$

For the second (again stationary) case, when $\alpha = p$ $(p = 1,2,3)$, one can obtain the following expression

$$\frac{d}{dt} S_p(u,Y) + \frac{1}{4} (u_p)^2 X_{,p} Y_p = 0, \tag{23}$$

where for the different $p = 1,2,3$ the functions $S_p(u,Y)$ are defined in the following way

$$\begin{aligned} S_p(u,Y) &:= (u_{p,1} u_1 + u_{p,2} u_2 + u_{p,3}) \\ &\quad - \frac{1}{2}\Big(u_1^2 Y_{1,p} + u_2^2 Y_{2,p} + u_3^2 Y_{3,p} \\ &\quad + u_{p-1} u_p Y_{p,p-1} + u_{p+1} u_p Y_{p,p+1}\Big). \end{aligned} \tag{24}$$

These equations, obtained from the generalized deviation equation $\nabla_u \nabla_u u = 0$ (14) for the case $\xi = u = v$, represent complicated *nonlinear differential equations* with respect to all the variables $X := \ln g_{00}$ and $Y_l := \ln g_{ll}$ $(l,m = 1,2,3)$. The first equation (21) for the $\alpha = 0$ component can be rewritten in the following form

$$\begin{aligned} &K_1(u_1,u_2,u_3,Y_1,Y_2,Y_3) X_{,1} \\ &\quad + K_2(u_1,u_2,u_3,Y_1,Y_2,Y_3) X_{,2} \\ &\quad + K_3(u_1,u_2,u_3,Y_1,Y_2,Y_3) X_{,3} = 0, \end{aligned} \tag{25}$$

where $K_1(..)$, $K_2(...)$ and $K_3(.....)$ are functions, depending on u_l and $Y_{l,k}$, which shall not be written here.

Our further aim will be to show that from the system (23) of differential equations in partial derivatives it will be possible to determine Y_l $\alpha = l = 1,2,3$ and after that, if substituted into (25), the solutions with respect to X can be found. As a result, the whole system of equations for all the variables X, Y_l will turn out to be integrable.

Since the variables X and Y_p do not depend on time, one can differentiate eq. (23) once more with respect to time (the time dependence in this equation is preserved in the u_p-terms and their derivatives) and from both the equations one can obtain an equation, no longer depending on the variable X

$$\frac{\dot{S}_p}{\ddot{S}_p} = -\frac{1}{2}\frac{u_p}{\dot{u}_p} \Rightarrow \left[\ln \dot{S}_p\right]^{\cdot} = -\left[\ln u_p\right]^{\cdot} . \qquad (26)$$

From here the solution for S_p is easily found after two consequitive integrations.This means that using the *assumed* property of *stationarity* of the metric tensor components, the above nonlinear system of equations (25) with respect to the variables X, Y_p has been reduced to a system of equations (26) in partial derivatives with respect to the variables Y_p.

Concerning the equation (25) for the $X = \ln g_{00}$ component, it is shown in [21] (see also Cartan's well-known monograph [22]) that *a necessary and sufficient condition* for the function $X(x,y,z)$ to be a *first integral,* i.e. to satisfy (25) is the fulfillment of the following system of differential equations

$$\frac{dx}{K_1(Y_p(x,y,z)} = \frac{dy}{K_2(Y_p(x,y,z)} = \frac{dz}{K_3(Y_p(x,y,z)} . \qquad (27)$$

Then for all the variables (x,y,z), satisfying the above equations, the function $X(Y_p(x,y,z))$ will be a first integral of equation (25), i.e. will be a constant.

CONCLUSIONS

In this paper a qualitatively new problem has been considered- finding the metric tensor components if a celestial mechanics trajectory is known. Only the deviation from a single trajectory was considered, since then the GDE acquires the most simple form $\nabla_u \nabla_u u = 0$. Even in such a simple form, the equations for the (diagonal) metric tensor components turn out to be complicated nonlinear equations. However, for the case near the vicinity of the libration points, the system of equations turns out to be integrable - first the g_{ii} components can be found from a system of differential equations, and subsequently the g_{00} component. It will be more interesting to consider the case for deviation from two vector fields - for the case of the Jovian 2002 experiment these can be the translational vector of motion and the angular motion vector. This will enable finding the light deflection from rotating bodies, which would require more precise astrometric measurements.

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ОБОБЩЕНО УРАВНЕНИЕ НА ДЕВИАЦИЯТА НА С. МАНОВ И ВЪЗМОЖНИТЕ МУ ПРИЛОЖЕНИЯ В НЕБЕСНАТА МЕХАНИКА И РЕЛАТИВИСТКАТА АСТРОМЕТРИЯ

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(Резюме)

Съвременната релативистка небесна механика (РНМ) води своето начало от работите на де-Ситтер от 1918 г., в които чрез методите на ОТО се извеждат формули за изменението на перихелия на Меркурий, Венера, Марс и Земята. Независимо от това и последващото развитие на РНМ (например монографиите на В.А. Брумберг), обратната задача за намиране на метрическия тензор на гравитационното поле при зададени параметри на небесно-механическите траектории и досега остава нерешена.

В доклада се обосновава и прилага нов подход за решаването на такава (обратна) задача чрез използването на т. н. "обобщено уравнение на девиация" (ОУД), предложено за първи път от проф. С. Манов (ИЯИЯЕ, БАН) през 1999 г. За разлика от известното уравнение за геодезична девиация в ОТО, което задава инфинитезимални отклонения от една геодезична линия, ОУД задава произволно (неинфинитезимално) отклонение от две произволно зададени (негеодезични) векторни полета (траектории) при зададени компоненти на метрическия тензор, което прави възможно приложението му в небесната механика

Предлаганият нов подход се основава на решаването на обратната задача, която условно би могла да се раздели на два етапа: 1. задаване на небесно-механичните траектории (векторни полета) посредством решенията на известната система уравнения на Хилл-Браун в рамките на т. н. "ограничена задача за движение на три тела Земя-Луна-Слънце" 2. заместване на тези решения в ОУД и решаването на съответните (в общия случай – нелинейни) диференциални уравнения относно компонентите на метрическия тензор при предположение за диагонална и стационарна метрика. Показано е, че за случая на либрационните точки върху траекториите на движение, където потенциалната енергия има екстремум, намирането на компонентите на метрическия тензор е възможно, тъй като получените уравнения могат да се интегрират.

Дискутират се накратко и възможните приложения на ОУД в релативистката астрометрия, предвид на свойството на ОУД да отчита относителното ускорение, породено от кривината на пространствено-временната геометрия при отклонението на светлинен лъч например в силно гравитационно поле. Стандартните уравнения на геодезичните и на геодезична девиация от ОТО не притежават това свойство, тъй като математическият им извод предполага закрепени краища на геодезичната линия, което се нарушава при наличие на относително ускорение.