Bulgarian Chemical Communications, Volume 46, Number 3(pp. 611– 615) 2014

Analytical solution of a transient Hartmann flow with Hall current and ion slip using finite Fourier transform

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Received August 8, 2013; Revised November 18, 2013

The transient Hartmann flow of an electrically conducting viscous incompressible fluid bound by two parallel insulating porous plates is studied using finite Fourier transform. An external uniform magnetic field is applied while the fluid motion is subjected to a constant pressure gradient. The Hall current and the ion slip are taken into consideration in the momentum equations. The effect of the Hall current and ion slip on the velocity and distribution of the flow is investigated.

Key words: Hartmann flow; Finite Fourier sine transform (FFST); Laplace transform (LT); conducting fluid; Hall current; Ion slip.

INTRODUCTION

The Hartmann flow defined as a magnetohydrodynamic (MHD) flow between two parallel plates is a classical issue that has many applications in MHD pumps, MHD power generators, accelerators, aerodynamics heating, and petroleum industry. In ref. [1] the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two insulated infinite parallel stationary plates is studied. Then a lot of research work has been done concerning the Hartmann flow under different physical effects [2-11]. In most cases the Hall and ion slip terms were ignored in applying Ohm's law as they have no marked effect for small and moderate values of the magnetic field. However, when an application of MHD requests a strong magnetic field, the influence of electromagnetic force becomes noticeable [4]. Under these conditions, the Hall current and ion slip are important as they will affect the magnitude and direction of the current density and consequently affect the magnetic force term. In ref. [6] the Hall effect on the steady motion of electrically conducting and viscous fluids in channels was studied. In [8-9] the effect on the steady MHD Couette flow with heat transfer was studied. The temperatures of the two plates were assumed either to be constant [8] or to vary linearly along the plates in the direction of the flow [9]. In [11] the effect of Hall current on the steady Hartmann flow subjected to a uniform suction and injection at the bounding plates was studied. Later, in [12] the problem of the unsteady state with heat transfer was extended taking into consideration the Hall effect while neglecting the ion slip. In [13] the ion slip was taken into consideration and the equations of motion were solved analytically using the Laplace transform (LT) method. The energy equation was solved numerically using the finite difference method taking into consideration the Joule and viscous dissipations.

In this paper, an analytical solution is presented for the transient flow of an incompressible, viscous, electrically conducting fluid between two infinite insulating horizontal plates with the consideration of both the Hall current and ion slip. The fluid is acted upon by a constant pressure gradient in the axial direction, while a uniform magnetic field is applied perpendicular to the plates. The induced magnetic field is neglected by assuming a very small magnetic Reynolds number. The momentum equations are solved analytically using Finite Fourier Sine transform (FFST). The effect of the magnetic field, the Hall current, and the ion slip on the velocity distribution is studied.

DESCRIPTION OF THE PROBLEM

The two insulating porous plates are located at $y=\pm h$ and extended from $x=0$ to ∞ and from $z=0$ to ∞ as shown in Fig.1.

The fluid flow between the two plates is influenced by a constant pressure gradient $\frac{dP}{dx}$ in

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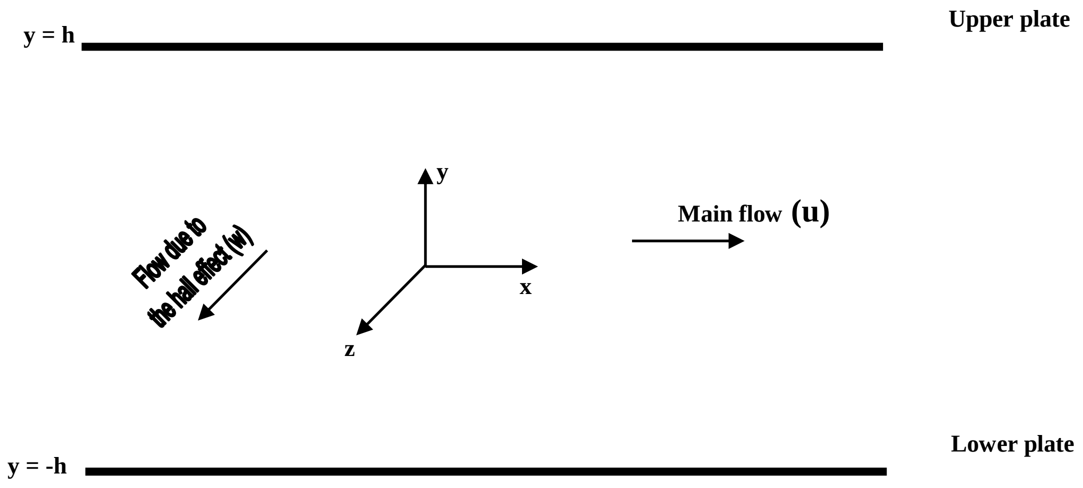

Fig. 1. Schematic diagram of the system.

the x-direction. The whole system is subject to a uniform magnetic field in the positive y-direction while the induced magnetic field is neglected. The existence of the Hall term gives rise to the z-component of the velocity. Thus the velocity vector of the fluid is given as:

$$\underline{v}(y,t) = u(y,t)\underline{i} + w(y,t)\underline{k} \tag{1}$$

The fluid flow is governed by the momentum equation

$$\rho \frac{Dv}{Dt} = \mu \nabla^2 v - \nabla P + J \wedge B_o \tag{2}$$

where ρ and μ are the density and the coefficient of viscosity of the fluid, respectively. If the Hall and ion slip terms are retained, the current density J is given by

$$J = \sigma \left\{ v \wedge B_o - \beta (J \wedge B_o) + \frac{\beta B_i}{B_o} (J \wedge B_o) \wedge B_o \right\} \tag{3}$$

where σ is the electric conductivity of the fluid, β is the Hall factor and B_i is the ion slip parameter [4]. Eqn. (3) may be solved in J to yield

$$J \wedge B_o = -\frac{\sigma B_o^2}{(1+\beta_i \beta_e)^2 + \beta_e^2} \{((1+\beta_i\beta_e)u + \beta_e w)i + ((1+\beta_i\beta_e)w - \beta_e u)k\} \tag{4}$$

where $\beta_e = \sigma \, \beta \, B_o$ is the Hall parameter [4]. Thus, in terms of Eqns. (1) and (4), the two components of Eqn. (2) read

$$\rho \frac{\partial u}{\partial t} = -\frac{dP}{dx} + \mu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_o^2}{(1+\beta_i\beta_e)^2 + \beta_e^2} ((1+\beta_i\beta_e)u + \beta_e w), \tag{5}$$

$$\rho \frac{\partial w}{\partial t} = \mu \frac{\partial^2 w}{\partial y^2} - \frac{\sigma B_o^2}{(1+\beta_i\beta_e)^2 + \beta_e^2} ((1+\beta_i\beta_e)w - \beta_e u), \tag{6}$$

The second and third terms on the right-hand side represent the viscous and Joule dissipations, respectively.

The problem is simplified by writing the equations in non-dimensional form. The characteristic length is taken to be h, and the characteristic time is $\frac{\rho h^2}{\mu^2}$ while the characteristic velocity is $\frac{\mu}{\rho h}$. We define the following non-dimensional quantities:

$$\hat{x} = \frac{x}{h}, \hat{y} = \frac{y}{h}, \hat{z} = \frac{z}{h}, \hat{u} = \frac{\rho h u}{\mu}, \hat{w} = \frac{\rho h w}{\mu}, \hat{P} = \frac{P \rho h^2}{\mu^2}, \hat{t} = \frac{t\mu}{\rho h^2},$$

In terms of the above non-dimensional variables and parameters, Eqns. (5)-(6) are written as (the "hats" will be dropped for convenience):

$$\frac{\partial u}{\partial t} = -\frac{dP}{dx} + \frac{\partial^2 u}{\partial y^2} - \frac{Ha^2}{(1+\beta_i\beta_e)^2 + \beta_e^2} ((1+\beta_i\beta_e)u + \beta_e w) \tag{7}$$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial y^2} - \frac{Ha^2}{(1+\beta_i\beta_e)^2 + \beta_e^2} ((1+\beta_i\beta_e)w - \beta_e u) \tag{8}$$

where Ha is the Hartmann number subjected to the following initial and boundary conditions:

$$u = w = 0 \;\; ; \;\; t \le 0$$
$$u = w = 0 \;\; ; \;\; y = \pm 1 \tag{9}$$

The main purpose of this paper is to solve the partial differential Eqns. (7), and (8) by FFST.

SOLUTION OF THE EQUATIONS OF MOTION USING FFST

The FFST of a function   is defined as:

$$F_s(g(y,t)) = G_s(n,t) = \int_0^L g(y)\sin(\frac{n\pi y}{L})\,dy \;;\; 0 \le y \le L \;;\; n = 1,2,3,...... \quad (10)$$

where the kernel of the transform is:

$$\sin\,(\frac{n\pi y}{L}) \quad (11)$$

For transforming the second derivative we use the operational property

$$F_s\left(\frac{d^2 g(y)}{dy^2}\right) = \frac{n\pi}{L}\left(g(0) - \cos(n\pi)g(L)\right) - (\frac{n\pi}{L})^2 G_s(n) \quad (12)$$

Thus, if the boundary value problem involves a second-order derivative, extends over a finite domain $0 \le y \le L$, and has boundary conditions at both ends in the following form:

$$g(0,t) = f_1(t) \qquad g(L,t) = f_2(t), \quad (13)$$

then FFST can be used to transform the 2nd-order derivative.

Defining:

$$v = u + iw \quad (14)$$

Eqns. (7) and (8) can be combined as:

$$\frac{\partial v}{\partial t} = -\frac{dP}{dx} + \frac{\partial^2 v}{\partial y^2} - \frac{Ha^2}{(1+\beta_i\beta_e)^2 + \beta_e^2}(1+\beta_i\beta_e - i\beta_e)v \quad (15)$$

Applying FFST to Eqn. (15) yields

$$\frac{\partial V_s}{\partial t} + (\frac{n^2\pi^2}{4} + k)V_s = \frac{dP}{dx} * \frac{2}{n\pi}(\cos n\pi - 1) \quad (16)$$

where

$$k = \frac{Ha^2}{(1+\beta_i\beta_e)^2 + \beta_e^2}\left(1+\beta_i\beta_e - i\beta_e\right) \quad (17)$$

The solution of the linear inhomogeneous partial differential Eqn. (16) under the initial and boundary conditions given in Eqn. (9) is given as

$$V_s(n,t) = \frac{\frac{dP}{dx} * \frac{2}{n\pi}(\cos n\pi - 1)}{(\frac{n^2\pi^2}{4} + k)}\left(1 - e^{-\left(k + \frac{n^2\pi^2}{4}\right)t}\right) \quad (18)$$

The inverse transform of and their real and imaginary components are given by

$$v(y,t) = \sum_{n=1}^{\infty}\left(\frac{\frac{dP}{dx} * \frac{2}{n\pi}(\cos n\pi - 1)}{\left(\frac{n^2\pi^2}{4} + k\right)}\left(1 - e^{-\left(k + \frac{n^2\pi^2}{4}\right)t}\right)\right)\sin\left(\frac{n\pi(y+1)}{2}\right) \quad (19)$$

$$u(y,t) = \mathrm{Re}\{v(y,t)\}, \quad w(y,t) = \mathrm{Im}\{v(y,t)\} \quad (20)$$

Figs. 2(a) and 2(b) present the evolution of the profiles of the velocity components u and w versus y for $H_a = 3$; $\beta_e = \beta_i = 3$; $\frac{dP}{dx} = -5$, and for various values of time t (0.1; 0.5; 1.5; 3.0). The figures show the parabolic shape of the profiles and indicate that both u and w reach their steady state monotonously with time.

RESULTS AND DISCUSSION

As seen in Tables 1 and 2, the comparison between LT as used by Attia [13] and FFST methods for solving the equation of motion for Hartmann flow indicates that the outputs are nearly equal which lends confidence to the results obtained here.

Table 1. Comparison between LT and FFST for calculating u

Y	*t*=0.5		*t*=1.5		*t*=3	
	LT	FFST	LT	FFST	LT	FFST
-1	0	0	0	0	0	0
-0.8	0.5815	0.5811	0.6898	0.6898	0.6936	0.6936
-0.6	0.9968	0.9961	1.2029	1.2028	1.2101	1.2101
-0.4	1.2734	1.2724	1.5571	1.5569	1.567	1.567
-0.2	1.4312	1.43	1.7647	1.7646	1.7764	1.7763
0	1.4825	1.4812	1.8331	1.833	1.8454	1.8453
0.2	1.4312	1.43	1.7647	1.7646	1.7764	1.7763
0.4	1.2734	1.2724	1.5571	1.5569	1.567	1.567
0.6	0.9968	0.9961	1.2029	1.2028	1.2101	1.2101
0.8	0.5815	0.5811	0.6898	0.6898	0.6936	0.6936
1	0	0	0	0	0	0

Table 2. Comparison between LT and FFST for calculating w

Y	*t*=0.5		*t*=1.5		*t*=3	
	LT	FFST	LT	FFST	LT	FFST
-1	0	0	0	0	0	0
-0.8	0.0229	0.0229	0.0438	0.0437	0.0456	0.0456
-0.6	0.0429	0.0428	0.0824	0.0824	0.0859	0.0859
-0.4	0.0579	0.0578	0.1124	0.1124	0.1172	0.1172
-0.2	0.0673	0.0671	0.1313	0.1313	0.1369	0.1369
0	0.0704	0.0703	0.1378	0.1377	0.1436	0.1436
0.2	0.0673	0.0671	0.1313	0.1313	0.1369	0.1369
0.4	0.0579	0.0578	0.1124	0.1124	0.1172	0.1172
0.6	0.0429	0.0428	0.0824	0.0824	0.0859	0.0859
0.8	0.0229	0.0229	0.0438	0.0437	0.0456	0.0456
1	0	0	0	0	0	0

In Fig. 2, the velocity component u reaches the steady state faster than w.

Fig. 3 presents the time evaluation of u and w at y=0 for various values of the Hall parameter β_e , the ion slip parameter β_i at $Ha = 1$.

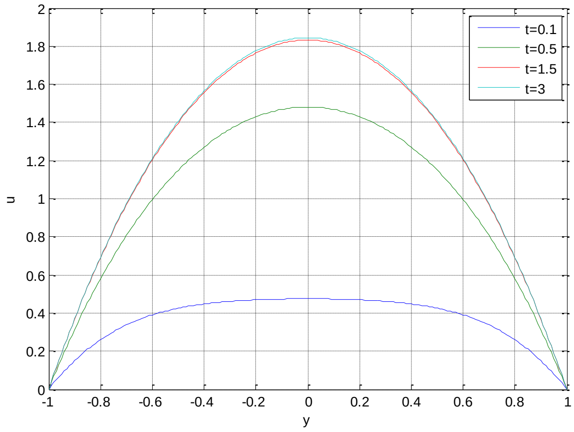

Fig. 2(a). Variation of u *versus* y.

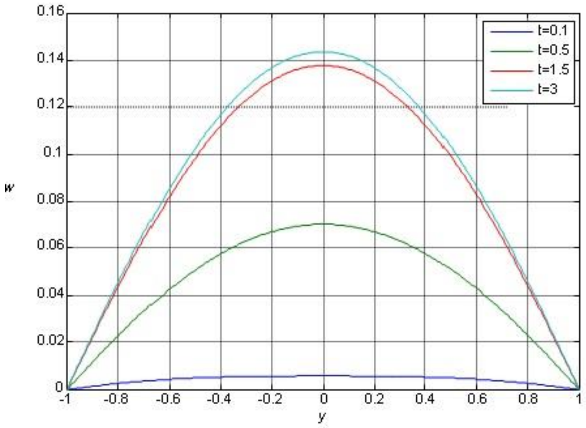

Fig. 2(b). Variation of w *versus* y.

In Fig. 3(a), increasing the parameters β_e and β_i will increase u because the effective conductivity $\frac{\sigma}{(1+\beta_i\beta_e)^2+{\beta_e}^2}$ decreases with increasing β_e or β_i which reduces the magnetic damping force on u.

In Fig. 3(b), increasing β_e will increase the velocity component w as a result of the Hall effect, but increasing β_i will decrease w for all values of β_e as a result of decreasing the source term of w ($\frac{\beta_e Ha^2 u}{(1+\beta_i\beta_e)^2+{\beta_e}^2}$) and increasing its damping term ($\frac{(1+\beta_i\beta_e)Ha^2 w}{(1+\beta_i\beta_e)^2+{\beta_e}^2}$), the influence of ion slip on w becoming clearer for higher values of β_e

In Fig. 3(b), increasing β_e will increase the velocity component w as a result of the Hall effect, but increasing β_i will decrease w for all values of β_e as a result of decreasing the source term of w ($\frac{\beta_e Ha^2 u}{(1+\beta_i\beta_e)^2+{\beta_e}^2}$) and increasing its damping term ($\frac{(1+\beta_i\beta_e)Ha^2 w}{(1+\beta_i\beta_e)^2+{\beta_e}^2}$), the influence of ion slip on w becoming clearer for higher values of β_e

For large β_e the components u and w overshoot, exceeding their steady state values and then go down towards steady state; the ion slip plays a role in suppressing these overshoots.

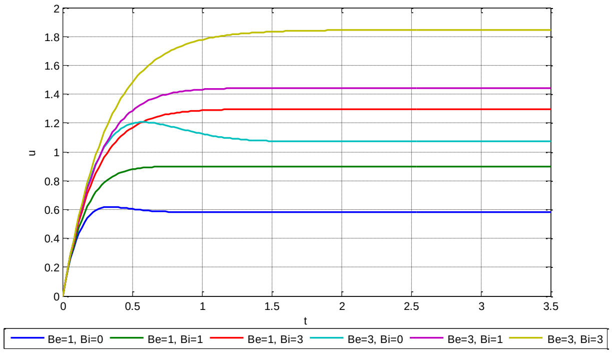

Fig. 3(a). Effect of Hall Parameter β_e and ion slip parameter β_i on the time development of u at y=0.

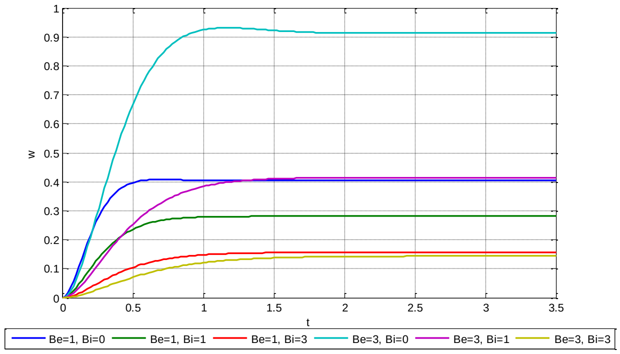

Fig. 3(b). Effect of Hall Parameter β_e and ion slip parameter β_i on the time development of w at y=0.

Fig. 4 presents u, and w at y=0 for various values of the Hartmann number Ha and the ion slip parameter β_i with $\beta_e = 3$.

As shown in Fig. 4(a), for small values of Ha, increasing β_i will slightly decrease u as a result of increasing the damping factor on u; further increasing β_i will increase the effective conductivity and, in turn, will decrease the damping factor on u which increases u; on the other hand, for larger values of Ha, u becomes small; increasing β_i always decrease the effective conductivity and consequently, will increase u, the effect of on u becoming more apparent for large values of Ha.

In Fig. 4(b), increasing the ion slip parameter β_i will decrease w for all values of Ha, its effect is more apparent for higher values of Ha.

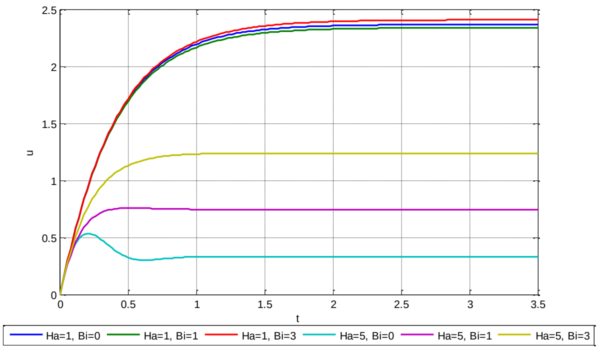

Fig. 4(a). Effect of Hartmann Number Ha and ion slip parameter β_i on the time development of u at y=0

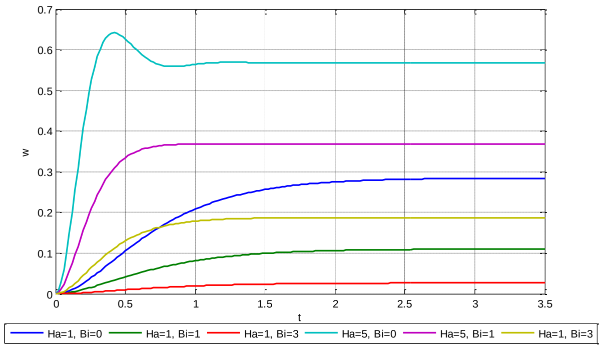

Fig. 4(b). Effect of Hartmann Number Ha and ion slip parameter β_i on the time development of w at Y=0.

CONCLUSIONS

FFST method can be used to obtain an analytical solution for the transient Hartmann flow of an electrically conducting, viscous, incompressible fluid bound by two parallel insulating plates with Hall current and ion slip. The comparison of the FFST method with previously used methods as LT shows that this technique is very simple and gives accurate results for solving the governing momentum equation for the whole range of the physical parameters used.

REFERENCES

1. J Hartmann and F Lazarus kgl. Danske Videnskab. Selskab Mat.-Fys. Medd. 15 (1937).
2. N Tao Journal of Aerospace Science 27 334 (1960).
3. S Nigam and S Singh The Quarterly Journal of Mechanics and Applied Mathematics 13 85 (1960).
4. K Cramer and S Pai (New York: McGraw-Hill) (1973).
5. G Sutton and A Sherman (New York: McGraw-Hill) (1965).
6. I Tani Journal of Aerospace Science 29 287 (1962).
7. H. A. Attia and M. A. M. Abdeen Effect of ion slip on the time-varying Hartmann flow of a non-Newtonian viscoelastic fluid with heat transfer, Journal of Applied Mechanics and Technical Physics 54 268 (March 2013)
8. V Soundalgekar, N Vighnesam and H Takhar IEEE Transactions on Plasma Science PS-7 178 (1979).
9. V Soundalgekar and A Uplekar IEEE Transactions on Plasma Science PS-14 579 (1986).
10. M. A. M. Abdeen, H. A. Attia, W. Abbas and W. Abd El-Meged Effectiveness of porosity on transient generalized Couette flow with Hall effect and variable properties under exponential decaying pressure gradient Indian Journal of Physics 87 767 (August 2013).
11. E Abo-El-Dahab Effect of Hall Currents on Some Magneto-hydrodynamic Flow Problems (Helwan University, Helwan: Master Thesis) (1993).
12. H Attia Canadian Journal of Physics 76 739 (1998).
13. H Attia Physica Scripta 66 470 (2002).

АНАЛИТИЧНО РЕШЕНИЕ НА ЗАДАЧАТА ЗА ПРЕХОДНО ТЕЧЕНИЕ НА HARTMANN С ТОК НА HALL И ЙОННО ПРИПЛЪЗВАНЕ СПОМОЩТА НА КРАЙНА ТРАНСФОРМАЦИЯ НА FOURIER

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Постъпила на 8 август, 2013 г.; коригирана на 18 ноември, 2013 г.

(Резюме)

Изследвано е преходното течение на Hartmann flow на електропроводящ несвиваем вискозен флуид между две успоредни изолаторни плочи с помощта на крайна Fourier'ова трансформация. Задачата е решена при прилагане на външно магнитно поле и постоянен градиент на хидравличното налягане. Токът на Hall и приплъзването на йони са отчетени в уравненията на движението. Установено е тяхното влияние върху скоростния профил в течността.