Bulgarian Chemical Communications, Volume 47, Special Issue B (pp. 269–274) 2015

Non-universal critical properties of the ferromagnetic mean spherical model with long-range interaction

E. S. Pisanova*, S. I. Ivanov

Department of Theoretical Physics, University of Plovdiv, 24 Tzar Assen Str., 4000 Plovdiv, Bulgaria

The bulk critical behavior of the mean spherical model with long-range interaction (decaying at large distances r as $r^{-d-\sigma}$, where d is the space dimensionality and $0<\sigma\leq 2$) is studied at the upper critical dimension by using the properties of the Lambert W-function. Exact expressions for the spherical field, the free energy density and the specific heat per spin are presented. The exact results are compared with the asymptotic ones on the basis of the calculated absolute and relative errors. Asymptotic analytical expressions for the absolute errors are also provided. It is shown that the obtained results are valid in a broader neighborhood of the critical point.

Key words: bulk critical behavior, upper critical dimension, long-range interaction, the Lambert W-function

INTRODUCTION

The spherical model of a ferromagnet of Berlin and Kac [1] is one of the few statistical mechanical models which have been exactly solved in any space dimensionality d and exhibit a non-trivial critical behavior for $d_l < d < d_u$, where d_l and d_u are the lower and the upper critical dimensions, respectively. It has been defined on the regular d-dimensional lattice. With each lattice site one associates a continuous real variable (spin). The spins of this model are subject to a global constraint with spherical symmetry, while those of the Ising model are subject to local ones. When the global constraint of Berlin and Kac [1] is satisfied in the sense of an ensemble average, the model is known as the mean spherical model. As it could be expected, the spherical model and the mean spherical model have the same thermodynamic properties (see [2]). The equivalence between the infinite translational invariant standard spherical model and the Heisenberg model with n spin components, in the case $n\to\infty$, has been discussed in [2–4].

The investigation of systems with long-range interaction was initiated on the ferromagnetic spherical model [5]. The spherical model has been extensively used for analytic exploration of the scaling properties of confined systems, in one or more directions (see e.g. [2, 6–11] and references therein). The Casimir effect which remains the central theme of both theoretical and experimental investigations, has been theoretically studied on the spherical model in several papers [12–16].

Different quantum versions of the spherical model (or the large n-limit of $O(n)$ symmetric models), some of which are models of Bose gas, are also available [17–27].

Recently it has been shown [28–30] that the Lambert W-function can be applied for an exact computation of non-universal critical properties with leading logarithmic behavior at the upper critical dimension of the system. The basic mathematical properties of the Lambert W-function, some of its applications as well as interesting historical remarks have been presented in [31].

In this paper, using the properties of the Lambert W-function we study the bulk critical behavior of the ferromagnetic mean spherical model with long-range interaction (decreasing at long distances r as $r^{-d-\sigma}$, $0<\sigma\leq 2$) at the upper critical dimension, $d=2\sigma$. At zero field in the neighborhood of the critical point, exact analytical results for the spherical field, the free energy density and the specific heat per spin are obtained and compared with the known asymptotic ones [2] on the basis of the calculated absolute and relative errors. Special attention is paid to the cases when the upper critical dimension coincides with the real physical dimensions (chains, thin layers, i.e. films and three-dimensional systems). It is shown that the obtained results hold true in a broader neighborhood of the critical point. Besides, the expansion of the critical region is estimated.

* To whom all correspondence should be sent:
katia_pisanova@abv.bg

THE MODEL

The Hamiltonian of the mean spherical model on a d-dimensional lattice is

$$H = -\frac{1}{2}\sum_{ij} J_{ij} s_i s_j + \frac{\mu}{2}\sum_i s_i^2 - h\sum_i s_i, \qquad (1)$$

where the spin at the i-th site, $s_i \equiv s_i(\mathbf{r})$, is a continuous real variable, J_{ij} is the interaction matrix between spins at sites i and j, and h is an ordering external magnetic field. The spherical field μ provides the spherical constraint

$$\sum_i \left\langle s_i^2 \right\rangle = N, \qquad (2)$$

where N is the total number of spins on the lattice and $\langle\cdots\rangle$ denotes the standard thermodynamic average computed with the Hamiltonian (1).

For long-range interaction potential

$$J(r) \sim r^{-(d+\sigma)}, \quad \sigma > 0 \quad \text{and} \quad r \to \infty, \qquad (3)$$

the long-wavelength (small $k \equiv |\mathbf{k}|$) leading asymptotic of its Fourier transform is $U(k) \sim k^\sigma$. The case $\sigma \geq 2$ corresponds to short-range interaction, i.e. in this case the universality class does not depend on σ. For $0 < \sigma < 2$, we have long-range interaction and the critical behavior depends on σ. The critical behavior of systems with long-range interaction in restricted geometry was discussed in [32].

At zero field, the thermodynamic limit form of the mean spherical constraint (2) is

$$\frac{dU_{d,\sigma}(\phi)}{d\phi} = K, \qquad (4)$$

where $K = \beta J$, $\beta = 1/(k_B T)$ (with the Boltzmann's constant $k_B = 1$), $\phi = \mu/J$ is the scaled spherical field and the function U is defined by

$$U_{d,\sigma}(\phi) = (2\pi)^{-d}\int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \ln(\phi + |\mathbf{k}|^\sigma). \quad (5)$$

The left hand side of (4) is a strictly decreasing function of $\phi \geq 0$ attaining its maximum at $\phi = 0$ if $d > \sigma$. The considered model undergoes a phase transition at the critical point K_c, determined by

$$K_c = \frac{dU_{d,\sigma}}{d\phi}(0), \quad d > \sigma. \qquad (6)$$

The bulk free energy density of the model has the following form [2]:

$$\beta f(K) = \frac{1}{2}(U_{d,\sigma}(\overline{\phi}) - K\overline{\phi}) + \frac{1}{2}\ln K - K, \qquad (7)$$

where $\overline{\phi}$ is the solution of (4).

Equations (7) and (4) provide the basis for studying the bulk critical behavior of the model. By differentiating twice the free energy density (7) with respect to the temperature, one obtains the zero-field specific heat per spin

$$c(K) := -K^2\frac{\partial^2}{\partial K^2}(\beta f(K)) = \frac{1}{2} + \frac{1}{2}K^2\frac{\partial\overline{\phi}}{\partial K}. \qquad (8)$$

Since ϕ decreases when K increases, see (4), and $\phi = 0$ for $K > K_c$, then the zero-field specific heat per spin keeps its maximum value $c(K) = 1/2$ for all $K \geq K_c$ and the Dulong-Petit low of classical thermodynamics holds for all $T \leq T_c$ [2].

SOME EXACT RESULTS AND THE ASYMPTOTIC ONES

The function U, defined by (5), can be represented in the form

$$U_{d,\sigma}(\phi) = \frac{S_d}{(2\pi)^d}\int_0^{x_D} x^{d-1}\ln(\phi + x^\sigma)dx, \qquad (9)$$

where $x_D = 2\pi(d/S_d)^{1/d}$ is the radius of the sphericalized Brillouin zone, $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the d-dimensional unit sphere and Γ is the Euler gamma function.

At the upper critical dimension $d = 2\sigma$ in the neighborhood of the critical point ($K_c/K - 1 \to 0^+$), i.e. when $\phi \ll 1$, the asymptotic behavior of the function (9) is

$$U_{2\sigma,\sigma}(\phi) \approx \left(\ln x_D^\sigma - \frac{1}{2}\right) + 2\left(\frac{\phi}{x_D^\sigma}\right) + \left(\frac{\phi}{x_D^\sigma}\right)^2 \ln\left(\frac{\phi}{x_D^\sigma}\right) - \frac{1}{2}\left(\frac{\phi}{x_D^\sigma}\right)^2, \qquad (10)$$

and for the mean spherical constraint (4), one obtains

$$\left(\frac{\phi}{x_D^\sigma}\right)\ln\left(\frac{\phi}{x_D^\sigma}\right) \approx -\left(\frac{K_c}{K} - 1\right). \qquad (11)$$

The critical value, determined by (6), is $K_c = 2/x_D^\sigma$. By the substitution $\ln(\phi/x_D^\sigma) = t$, the equation (11) takes the form of the defining equation for the Lambert W-function [31]

$$W(x)e^{W(x)} = x$$

which always has an infinite number of solutions, most of them complex, and so $W(x)$ is a multivalued function. When x is a real variable, then for $-1/e \leq x < 0$ there are two branches with real values of $W(x)$: the branch $W_0(x)$ satisfying $-1 \leq W_0(x)$ and the branch $W_{-1}(x) \leq -1$, as $\lim_{x\to 0^-} W_{-1}(x) = -\infty$.

Thus, in therms of the Lambert W-function, the exact solution of (11) is

$$\overline{\phi} = x_D^\sigma \exp[W_{-1}(-t)], \tag{12}$$

where the variable $t = K_c/K - 1$ is a measure of the deviation from the critical point K_c. The choice of the real branch $W_{-1}(x)$ of the Lambert W-function among the solutions of (11) corresponds to the fact that at $t = 0\,(K = K_c)$ the scaled spherical field ϕ vanishes.

For the free energy density near the critical point, from (7), (10) and (12) omitting the terms of order $O(\overline{\phi}^2 \ln \overline{\phi})$, we obtain

$$\beta f(t) \approx \beta_c f(0) + \beta \delta f(t), \tag{13}$$

where the free energy density at $t = 0\,(K = K_c)$ is

$$\beta_c f(0) = \frac{1}{2}\left(\ln x_D^\sigma - \frac{1}{2}\right) + \frac{1}{2}\ln K_c - K_c \tag{14}$$

with correction

$$\delta(\beta f(t)) = \frac{1}{2} K_c t\, \overline{\phi} = t \exp[W_{-1}(-t)]. \tag{15}$$

For the specific heat per spin near the critical point from (8) and (12), using the derivative of the Lambert W-function, we get the following exact result

$$c(t) = \frac{1}{2} + \frac{1}{1 + W_{-1}(-t)}. \tag{16}$$

The last result shows that the specific heat capacity remains finite at $t = 0\,(K = K_c)$ but at this point its graph has a cusp. The obtained expression (16) allows us to find a critical region $(0, t_0]$ in which the zero-field specific heat per spin decreases to zero and at the endpoint, we have $c(t_0) = 0$. From this and (16), we obtain

$$W_{-1}(-t_0) = -3.$$

Thus, for the endpoint of the critical region, we get $t_0 = 3e^{-3}$.

Using the absolutely convergent series [31]

$$W_{-1}(-t) = L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2+L_2)}{2L_1^2} + \frac{L_2(6 - 9L_2 + 2L_2^2)}{6L_1^3} + \frac{L_2(-12 + 36L_2 - 22L_2^2 + 3L_2^3)}{12L_1^4} + O\left(\left(\frac{L_2}{L_1}\right)^5\right), \tag{17}$$

where $L_1 = \ln t$ and $L_2 = \ln(-\ln t)$, we get the asymptotic expressions for the scaled spherical field $\overline{\phi}$, the correction to the free energy density at $t = 0\,(K = K_c)$ and the specific heat per spin. Retaining the leading two terms of (17) from (12), (15) and (16), we obtain as follows

$$\overline{\phi}_{appr.} \approx -x_D^\sigma \frac{t}{\ln t}, \tag{18}$$

$$\delta(\beta f(t))_{appr.} \approx -\frac{t^2}{\ln t}, \tag{19}$$

and

$$c_{appr.}(t) \approx \frac{1}{2} + \frac{1}{\ln t}. \tag{20}$$

In the theory of phase transitions these asymptotic expressions show a typical behavior at the upper critical dimension. The logarithmic solution (18) is well known (see e.g. [2, p. 88]. It was obtained, neglecting the linear ϕ-term in (11). From (20), for the endpoint of the interval $(0, t_{0appr.}]$, we obtain $t_{0appr.} = e^{-2}$. The specific heats (16) and (20), are graphically presented in Fig. 1.

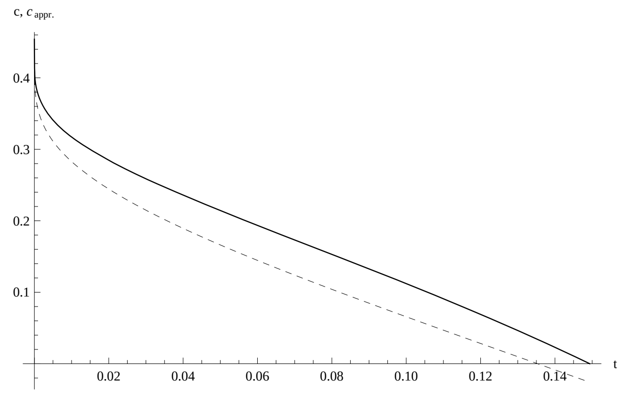

Fig. 1. The dependences of the specific heats c and $c_{appr.}$ on the deviation from the critical point t. The dashed line corresponds to $c_{appr.}$.

COMPARISON BETWEEN THE EXACT RESULTS AND THE ASYMPTOTIC ONES

From (17), retaining the leading L_1/L_2-term, we get the following asymptotic expressions for the absolute errors:

$$\Delta\overline{\phi} = |\overline{\phi} - \overline{\phi}_{appr.}| \approx x_D^{\sigma} \frac{t}{\ln^2 t}, \tag{21}$$

$$\Delta(\beta f) = \beta |f - f_{appr.}| \approx \frac{t^2}{\ln^2 t} \tag{22}$$

and

$$\Delta c = |c - c_{appr.}| \approx \frac{1}{2|\ln t|^3}. \tag{23}$$

On the other hand, we get the following estimate for the expansion of the critical region

$$\Delta t = t_0 - t_{0appr.} = (3 - e)/e^3. \tag{24}$$

For more detailed analysis, some numerical data for the absolute errors and relative errors are given in Table 1 and Table 2, respectively. Special attention is paid to systems with real physical dimensions (chains, thin layers, i.e. films and three-dimensional systems).

CONCLUSIONS

In a broader neighborhood of the critical point, the critical behavior of the mean spherical model with long-range interaction (decaying at large distances r as $r^{-d-\sigma}$, where d is the space dimensionality and $0 < \sigma \leq 2$) is studied.

Exact expressions for the scaled spherical field (12), the correction to the free energy density (15) and the specific heat per spin (16), in terms of the Lambert W-function are obtained. These results are compared with the known asymptotic ones on the basis of the calculated absolute and relative errors. Besides, using the series expansion of the Lambert W-function, asymptotic analytical expressions for the absolute errors are presented (see (21), (22) and (23)).

The obtained expressions (16) and (20) for the specific heat capacity per spin allow us to estimate the expansion of the critical region (24). This expansion easily can be seen in Fig. 1.

Let us note that the absolute error $\Delta\overline{\phi}$ and the relative error $|\Delta(\beta f)/(\beta f)|$ depend on both the deviation from the critical point t and the upper critical dimension $d = 2\sigma$ of the system, while the absolute errors $\Delta(\beta f)$ and Δc and the relative errors $|\Delta\overline{\phi}/\overline{\phi}|$ and $|\Delta c/c|$ depend only on t (see Table 1 and Table 2).

Finally, this treatment by using the Lambert W-function is applicable in a broader neighborhood of the critical point. Moreover, it can be applied to a wide class of models with leading logarithmic behavior at the upper critical dimensions.

Table 1. Numerical data for the absolute errors

		$\Delta\overline{\phi}$		$\Delta(\beta f)$	Δc
t	$d = 1(\sigma = 1/2)$	$d = 2(\sigma = 1)$	$d = 3(\sigma = 3/2)$		
1×10^{-5}	2.881×10^{-7}	5.762×10^{-7}	1.250×10^{-6}	1.625×10^{-12}	0.010
1×10^{-4}	4.052×10^{-6}	8.104×10^{-6}	1.759×10^{-5}	2.286×10^{-10}	0.014
1×10^{-3}	6.219×10^{-5}	1.243×10^{-4}	2.700×10^{-4}	3.509×10^{-8}	0.021
1×10^{-2}	1.110×10^{-3}	2.221×10^{-3}	4.821×10^{-3}	6.265×10^{-6}	0.034
1×10^{-1}	0.027	0.054	0.119	1.547×10^{-3}	0.046

Table 2. Numerical data for the relative errors

		$\lvert\Delta(\beta f)/(\beta f)\rvert[\%]$		$\lvert\Delta\overline{\phi}/\overline{\phi}\rvert[\%]$	$\lvert\Delta c/c\rvert[\%]$
t	$d = 1(\sigma = 1/2)$	$d = 2(\sigma = 1)$	$d = 3(\sigma = 3/2)$		
1×10^{-5}	1.575×10^{-10}	3.476×10^{-10}	9.952×10^{-10}	23.023	2.568
1×10^{-4}	2.215×10^{-8}	4.889×10^{-8}	1.399×10^{-7}	26.674	3.649
1×10^{-3}	3.401×10^{-6}	7.504×10^{-6}	2.148×10^{-5}	31.996	5.727
1×10^{-2}	6.072×10^{-4}	1.339×10^{-3}	3.836×10^{-3}	40.554	10.849
1×10^{-1}	0.150	0.332	0.963	55.353	41.321

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НЕУНИВЕРСАЛНИ КРИТИЧНИ СВОЙСТВА НА СРЕДНОСФЕРИЧНИЯ МОДЕЛ НА ФЕРОМАГНЕТИК С ДАЛЕКОДЕЙСТВИЕ

Е. Писанова, С. Иванов

Физически факултет, Пловдивски университет "Паисий Хилендарски", ул. "Цар Асен" №24, 4000 Пловдив, България

(Резюме)

Разгледан е класически средносферичен модел с далекодействие [1] в горната критична размерност. На основата на свойствата на функцията на Ламберт [2] са получени точни резултати за сферичното поле, плътността на свободната енергия и специфичната топлина (отнесена към един спин) в една по-широка околност на критичната точка. За случаите, когато горната критична размерност на системата съвпада с реалните физични размерности (верижка, тънък слой и тримерна система) получените точни резултати са сравнени с асимптотичните такива на база пресметнатите абсолютни и относителни грешки.

Разглеждането е приложимо към широк клас модели за точно пресмятане на неуниверсални критични свойства с асимптотично логаритмично поведение в горната критична размерност.

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